**NUMERICAL METHODS   
FOR MATHEMATICAL PHYSICS INVERSE PROBLEMS**

## Lecture 6. Inverse problems for linear stationary systems

We know that the inverse problems can be transformed to the problems of finding of extremum. So the practical methods of inverse problems theory are based on the optimization methods. The problems of the minimization of the functionals can be solved by means of the gradient methods. We applied it for the cases with direct dependence of the functional from unknown parameter of the system. However for the standard optimization control problems this dependence is not direct. In really the functional depends from the state function; and the state function depends from unknown parameter (control) by the state equation. It is true for optimization control problems, which are the transformation of mathematical physics inverse problems. So we will try to extend the known minimization methods to the optimization control problems.

### 6.1. Problem statement

We considered before the problem of finding a system parameter *v* such that the state function *y*, which is the solution of the equation

 (6.1)

satisfies the additional condition

 (6.2)

where *A* is a linear continuous operator from the unitary space *Y* to the unitary space *W*, *B* is a linear continuous operator from the unitary space *V* to *W*, *f* is given function of *W*, *C* is linear operator from *Y* to the unitary space *Z*, *z* is given function from *Z* (result of measuring). We transformed this inverse problem to the problem of the minimization of the functional



where  is the solution of the equation (6.1) for the fixed parameter *v*.

This optimization control problem can be solved with using the gradient method

 (6.3)

where  is a positive iterative parameter. The derivative of the functional here is determined by the formula

 (6.4)

where *pk* is the solution of the adjoint equation

 (6.5)

and 

We consider now a concrete inverse problem for the stationary system. Let Ω be   
*n*-dimensional set with the boundary *S*. The state of the system is described by Poisson equation

 (6.6)

with homogeneous boundary condition

 (6.7)

Consider the easiest measuring condition

 (6.8)

where *z* is a known function. We have the following inverse problem:

**Problem 1**. *Find the parameter v such that the solution y of Dirichlet problem* (6.6), (6.7) *satisfies the equality* (6.8).

This is inverse problem with distributed parameter and distributed measuring. It is not interesting in really because we can find  by equality (6.6), (6.8). However we will try to solve it by means of the gradient methods before the consideration of more interesting inverse problems.

### 6.2. Solving of the inverse problem with distributed parameter and distributed measuring

Let operator *A* from the abstract state equation (6.1) be Laplace operator, *B* and *C* are unit operators, and *f* is zero function. Let the state space *Y* be the space of the smooth enough functions with zero values on the boundary *S*. So the boundary problem (6.6), (6.7) can be transformed to the abstract operator equation (6.1). Besides the equality (6.8) has the form (6.2). Then we determine the functional *I* by the formula



where  is the solution of Dirichlet problem (6.6), (6.7) for the concrete parameter *v*.

…

### 6.3. Inverse problem with distributed parameter and pointwise measuring

We consider the equation (6.1) with distributed parameter and pointwise measuring

 (6.9)

where  are given points of the set Ω (points of measuring), and  are given numbers (results of measuring). In this case the minimizing functional will be determined by the formula



δ-function

…

### 6.4. Inverse problem with boundary parameter and boundary measuring

Let the boundary *S* of the given set Ω consists of two part *S*1 and *S*2. We consider Poisson equation

 (6.10)

where the function *f* is given. We do not have completely information on the boundary *S*. We have following boundary condition

 (6.11)

 (6.12)

where the function *g* is given, and the function *v* is unknown parameter. The additional condition is determined by the equality

 (6.13)

where *z* is given function (result of measuring), and the term at the left side of this equality is the normal derivative of the state function *y*. Thus we have two conditions (6.11) and (6.13) on the part *S*1 of the boundary. But we do not any condition on the *S*2.

Transform this inverse problem to the optimization control problem. Determine the functional



So we have the problem of the minimization of the functional *I*, where *y*[*v*] is the solution of the equation (6.10) with boundary conditions (6.11), (6.12).

Find the derivative of the functional at the point *v*. Determine the difference

 (6.14)

where *σ* is a constant, *h* is an arbitrary function on the set *S*2,  The function is the solution of the boundary problem

 (6.15)

 (6.16)

 (6.17)

Multiply the equality (6.15) to the arbitrary function *p* and integrate the result with respect to *x* of the set Ω. We obtain



Using second Green’s formula, we get



This equality can be transform to

 (6.18)

because of the boundary conditions (6.16), (6.17).

The equality (6.18) is true for all function *p*. Let *p* is the solution of the boundary problem

 (6.19)

 (6.20)

 (6.21)

Then the equality (6.18) is transformed to



So we get

 (6.22)

Estimate the second term at the right side of this equality. Multiply the equality (6.15) to the arbitrary function *p* and integrate the result with respect to *x* of the set Ω. We obtain



Using first Green’s formula, we get



because of the equalities (6.16), (6.17).

**Task 1**

Find the derivative and determine the gradient method for the following problems with unknown function *v*:

1) 

2) 

3) 

**Task 2**

Find the derivative and determine the gradient method for the following problems with unknown function *v*:

1) 

2) 

3) 